AIAA JOURNAL
Vol. 40, No. 7, July 2002

Transcendental Eigenvalue Problem and Its Applications

Kumar Vikram Singh* and Yitshak M. Ram†
Louisiana State University, Baton Rouge, Louisiana 70803

The algebraic eigenvalue problem is frequently encountered when analyzing the behavior of a multi-degree-of-freedom dynamic system. The characteristic equation associated with the algebraic eigenvalue problem is a polynomial that defines the eigenvalues by its roots. Dynamics and stability of distributed parameter systems are characterized by transcendental eigenvalue problems, with transcendental characteristic equations. By the use of finite element or finite difference methods, the transcendental eigenvalue problem is transformed to an algebraic problem. Because the behavior of a finite dimensional polynomial is fundamentally different from a transcendental function, such an approach may involve an inaccurate solution, which is attributed to a discretization error. The main idea is to replace the continuous system with variable physical parameters by a continuous system with piecewise uniform properties. The matching conditions between the various parts of the continuous model are expressed as a transcendental eigenvalue problem, which is then solved by the Newton's eigenvalue iteration method. Some classical problems in structural dynamics and stability are solved to demonstrate the method and its application.

I. Introduction

BY the transcendental eigenvalue problem, we denote the problem of determining the nontrivial solutions ω and $z \neq \mathbf{0}$ of

$$A(\omega)z = \mathbf{0} \tag{1}$$

where the elements of the $n \times n$ matrix A are transcendental functions in ω and z is a constant eigenvector. Such problems arise naturally in applications of mathematical physics. To demonstrate the motivation for this study, consider a piecewise uniform axial vibrating rod, such as that shown in Fig. 1, with axial rigidity

$$p(x) = \begin{cases} \alpha, & 0 < x < 0.5 \\ \beta, & 0.5 < x < 1 \end{cases} \tag{2}$$

and mass per unit length

$$q(x) = \begin{cases} \gamma, & 0 < x < 0.5 \\ \delta, & 0.5 < x < 1 \end{cases} \tag{3}$$

where α, β, γ, and δ are constants. The natural frequencies and mode shapes of the rod are determined by the Sturm–Liouville system of equations

$$\begin{aligned} &\alpha u'' + \omega^2 \gamma u = 0, && 0 < x < 0.5 \\ &\beta v'' + \omega^2 \delta v = 0, && 0.5 < x < 1 \\ &k u(0) = \alpha u'(0), && u(0.5) = v(0.5) \\ &\alpha u'(0.5) = \beta v'(0.5), && v'(1) = 0 \end{aligned} \tag{4}$$

where primes denote derivatives with respect to x. The four boundary and matching conditions in Eq. (4) stand for a spring force that is proportional to the displacement at $x = 0$, continuous displacement and continuous axial force at $x = 0.5$, and the stress-free end at $x = 1$. The general solutions for $u(x)$ and $v(x)$ are given by

$$u = z_1 \sin \omega x + z_2 \cos \omega x \tag{5}$$

$$v = z_3 \sin \omega x + z_4 \cos \omega x \tag{6}$$

where z_i, $i = 1, 2, 3, 4$, are constants. Noting that $\omega = 0$ is not a natural frequency of the rod, the matching and boundary conditions in Eq. (4) can be written in matrix form:

$$\begin{bmatrix} \alpha\omega & -k & 0 & 0 \\ \sin(\omega/2) & \cos(\omega/2) & -\sin(\omega/2) & -\cos(\omega/2) \\ \alpha\cos(\omega/2) & -\alpha\sin(\omega/2) & -\beta\cos(\omega/2) & \beta\sin(\omega/2) \\ 0 & 0 & \cos\omega & -\sin\omega \end{bmatrix} \times \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

by virtue of Eqs. (5) and (6). Problem (7), thus, has the transcendental eigenvalue problem form (1). We wish to find the eigenvalues ω_i that make the matrix $A(\omega_i)$ singular. Once an eigenvalue ω_i is determined, the problem of evaluating the corresponding eigenfunction

$$\phi_i(x) = \begin{cases} u(x, \omega_i), & 0 < x < 0.5 \\ v(x, \omega_i), & 0.5 < x < 1 \end{cases} \tag{8}$$

is reduced to the linear problem of solving Eq. (7) for z_i, $i = 1, 2, 3, 4$, and substituting these values in Eqs. (5) and (6).

In Sec. II we present a Newton's eigenvalue iteration method, originally developed by Yang,[1] for solving the transcendental eigenvalue problem. Examples involving vibration of rods and composite buckling of columns are then solved in Sec. III to demonstrate the method and its application. In Sec. IV, we develop a higher-order approximation technique, yielding an accelerated convergence. In Sec. V, we apply the method to determine the solutions of the quadratic eigenvalue problem without reduction to the first-order form, which involves doubling the size of the system. Through an example associated with vibrations of an exponential rod, we demonstrate in Sec. VI the use of the algorithm in estimating the spectrum of a nonuniform distributed system, and compare the results to that obtained by standard finite difference and finite element methods.

Received 5 September 2001; revision received 27 February 2002; accepted for publication 5 March 2002.

*Research Assistant, Mechanical Engineering Department.
†Professor, Mechanical Engineering Department.

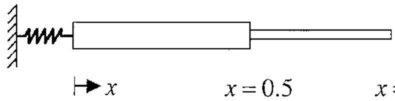

Fig. 1 Axial vibrating rod with piecewise uniform cross-sectional area.

II. Newton's Eigenvalue Iteration Method

Following Yang,[1] we denote

$$B(\omega) = -\frac{dA(\omega)}{d\omega} \tag{9}$$

and obtain, using a Taylor's series expansion of $A(\omega)$ about $\omega^{(0)}$,

$$A\left(\omega^{(0)} + \varepsilon\right) = A\left(\omega^{(0)}\right) - \varepsilon B\left(\omega^{(0)}\right) + \mathcal{O}(\varepsilon^2) \tag{10}$$

where $\mathcal{O}(\varepsilon^2)$ includes terms of order ε^2 and higher. Ignoring the second- and higher-order terms in ε gives

$$\det\left[A\left(\omega^{(0)} + \varepsilon\right)\right] = \det\left[A\left(\omega^{(0)}\right) - \varepsilon B\left(\omega^{(0)}\right)\right] \tag{11}$$

Hence, in the neighborhood of an eigenvalue ω of Eq. (1)

$$\omega = \omega^{(0)} + \varepsilon \tag{12}$$

and we have

$$\det\left[A\left(\omega^{(0)}\right) - \varepsilon B\left(\omega^{(0)}\right)\right] = 0 \tag{13}$$

It, thus, follows from Eq. (1) that

$$\left[A\left(\omega^{(0)}\right) - \varepsilon B\left(\omega^{(0)}\right)\right]z = 0 \tag{14}$$

The crucial point observed from Eq. (14) is that ε can be determined by the solution of eigenvalue problem (14). Hence, starting with an initial guess $\omega^{(0)}$, we may determine by an iterative manner in the ith iteration a new approximation:

$$\omega^{(i)} = \omega^{(i-1)} + \varepsilon^{(i)} \tag{15}$$

where $\varepsilon^{(i)}$ is an eigenvalue of

$$\left[A\left(\omega^{(i-1)}\right) - \lambda B\left(\omega^{(i-1)}\right)\right]z^{(i)} = 0 \tag{16}$$

For a given $\omega^{(i-1)}$, there are generally n eigenvalues λ_i, $i = 1, 2, \ldots, n$, where n is the dimension of the system. Because the Taylor approximation (10) is valid only for small ε, we will choose $\varepsilon^{(i)}$ as the smallest eigenvalue of Eq. (16) in an absolute value sense.

Suppose that the process has been converged after m iterative steps. Then the eigenvector $z = z^{(m)}$ obtained by the solution of Eq. (16) determines the eigenfunction ϕ_i. For example, in the vibrating rod of Sec. I the scalars z_i, $i = 1, 2, 3, 4$, in Eq. (7) determine the associated eigenfunction ϕ_i via Eqs. (5) and (6).

Let $n = 1$, and denote for this case $f(\omega) = A(\omega)$. Then Eq. (16) reduces to

$$\lambda = -\frac{f\left(\omega^{(i-1)}\right)}{f'\left(\omega^{(i-1)}\right)} \tag{17}$$

by virtue of Eq. (9). The new value for ω is determined by Eq. (12), with $\varepsilon = \lambda$. Thus, it follows that the Newton's method for evaluating the roots of a function is a degenerated one-dimensional case of the eigenvalue extraction algorithm.

III. Examples

We now present two examples demonstrating the method.

Axial Vibration of a Rod

Consider the transcendental eigenvalue problem associated with the nonuniform rod discussed in Sec. I. Let $k = 300$, $\alpha = 2$, $\beta = 1$, $\gamma = 2$, and $\delta = 1$. Then, using Eq. (7), we obtain

$$A(\omega) = \begin{bmatrix} 2\omega & -300 & 0 & 0 \\ \sin(\omega/2) & \cos(\omega/2) & -\sin(\omega/2) & -\cos(\omega/2) \\ 2\cos(\omega/2) & -2\sin(\omega/2) & -\cos(\omega/2) & \sin(\omega/2) \\ 0 & 0 & \cos\omega & -\sin\omega \end{bmatrix} \tag{18}$$

and, from Eq. (9),

$$B(\omega) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ -0.5\cos(\omega/2) & 0.5\sin(\omega/2) & 0.5\cos(\omega/2) & -0.5\sin(\omega/2) \\ \sin(\omega/2) & \cos(\omega/2) & -0.5\sin(\omega/2) & -0.5\cos(\omega/2) \\ 0 & 0 & \sin\omega & \cos\omega \end{bmatrix} \tag{19}$$

We have applied the algorithm of Sec. II to this problem with various initial values and determined the lowest five natural frequencies of the rod. The results are summarized in Table 1. The initial guess, the number of iterations applied, and the natural frequencies obtained (with precision tolerance of 10^{-12}) are shown in Table 1.

Table 1 Natural frequencies of the rod

i	$\omega_i^{(0)}$	Number of iterations	ω_i	$\det[A(\omega_i)]$
1	0.1	14	1.89795	1.81×10^{-16}
2	3.2	10	4.34375	3.02×10^{-16}
3	6.4	13	8.13909	3.33×10^{-16}
4	9.6	6	10.58616	9.66×10^{-16}
5	12.8	9	14.37982	9.06×10^{-17}

Composite Buckling of a Column

Consider the problem of finding the critical load P applied to a uniform column of unit length $L = 1$ and flexural rigidity EI. Suppose that an intermediate axial load Q is applied to the rod at $x = a$, as shown in Fig. 2a. Let $y(x)$ be the lateral deflection of the column, and denote $y_a = y(a)$. The free-body diagram of the entire column shown in Fig. 2b determines the forces at the supported ends. The free-body diagrams of the virtual cross sections shown in Figs. 2c and 2d imply that

$$-EIy'' = (P + Q)y - Qy_a x, \qquad 0 < x < a \tag{20}$$

$$-EIy'' = Py + Qy_a(1 - x), \qquad a < x < 1 \tag{21}$$

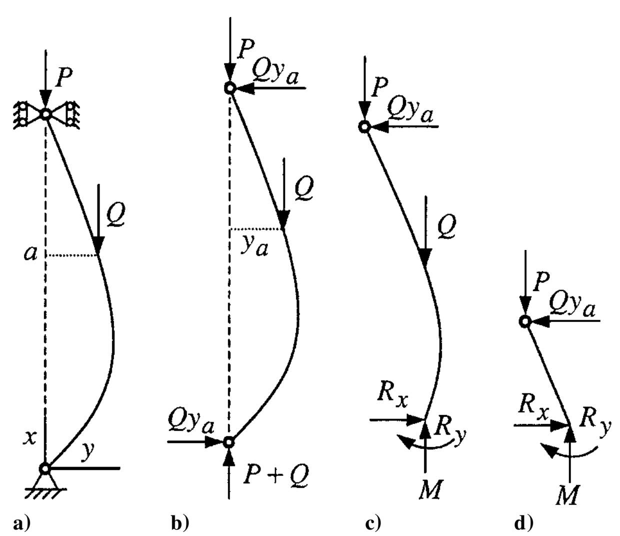

Fig. 2 Buckling of column with composite loads.

Denote

$$\sigma = \frac{P}{EI}, \qquad \mu = \frac{Q}{EI}, \qquad y(x) = \begin{cases} u(x) & 0 < x < a \\ v(x) & a < x < 1 \end{cases} \tag{22}$$

Then Eqs. (20) and (21) yield

$$u'' + (\sigma + \mu)u = \mu u_a x, \qquad 0 < x < a \tag{23}$$

$$v'' + \sigma v = -\mu v_a (1 - x), \qquad a < x < 1 \tag{24}$$

where $u_a = u(a)$ and $v_a = v(a)$. Thus, it follows that

$$u(x) = A_1 \sin \omega_1 x + A_2 \cos \omega_1 x + Cx \tag{25}$$

$$v(x) = B_1 \sin \omega_2 x + B_2 \cos \omega_2 x + D(1 - x) \tag{26}$$

where

$$\omega_1 = \sqrt{\sigma + \mu} \tag{27}$$

$$\omega_2 = \sqrt{\sigma} \tag{28}$$

$$C = \mu u_a / \sigma + \mu \tag{29}$$

$$D = -\mu v_a / \sigma \tag{30}$$

By imposing the boundary condition $u(0) = 0$, we obtain

$$A_2 = 0 \tag{31}$$

so that

$$u = A_1 \sin \omega_1 x + Cx \tag{32}$$

From the condition $v(1) = 0$, we obtain

$$B_1 \sin \omega_2 + B_2 \cos \omega_2 = 0 \tag{33}$$

The continuity of displacement and slope at $x = a$ imply

$$A_1 \sin \omega_1 a + Ca - B_1 \sin \omega_2 a - B_2 \cos \omega_2 a - D(1 - a) = 0 \tag{34}$$

$$A_1 \omega_1 \cos \omega_1 a + C - B_1 \omega_2 \cos \omega_2 a + B_2 \omega_2 \sin \omega_2 a + D = 0 \tag{35}$$

Equations (33–35), (29), and (30) can be written in the following matrix form:

$$\begin{bmatrix} 0 & \sin\omega_2 & \cos\omega_2 & 0 & 0 \\ \sin\omega_1 a & -\sin\omega_2 a & -\cos\omega_2 a & a & a-1 \\ \omega_1 \cos\omega_1 a & -\omega_2 \cos\omega_2 a & \omega_2 \sin\omega_2 a & 1 & 1 \\ 0 & 0 & 0 & \omega_1^2 & \omega_2^2 \\ \left(\omega_1^2 - \omega_2^2\right)\sin\omega_1 a & 0 & 0 & (a-1)\omega_1^2 - a\omega_2^2 & 0 \end{bmatrix} \begin{pmatrix} A_1 \\ B_1 \\ B_2 \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{36}$$

We note that ω_1 is determined by ω_2 via Eq. (28). Hence, Eq. (36) has the transcendental eigenvalue form with $\boldsymbol{A}(\omega_2)\boldsymbol{z} = \boldsymbol{0}$.

The Newton's eigenvalue method has been applied to this problem with variable parameters $a = 0.1k$, $k = 0, 1, \ldots, 10$, and $\hat{Q} = Q/EI = j$, $j = 1, 3, \ldots, 7$. The critical load $\hat{P} = P(\hat{Q}, a)/EI$ obtained is indicated by the graphs in Fig. 3. Timoshenko and Gere (Ref. 2, Sec. 2.11) have solved this problem before. Their results (see Table 2.6 of Ref. 2), displayed in Table 2, indicate complete agreement with the parametric solution obtained here.

Table 2 Comparison of the critical loads for $a = 0.5$

Values from Timoshenko and Gere[2]		Transcendental eigenvalue solution
$\hat{Q}$	$\hat{P}$	$\hat{P}$
0	9.8696	9.8696
2.1872	8.7487	8.7687
3.9728	7.9456	7.9590
5.3400	7.1200	7.1558
6.5197	6.5197	6.5444

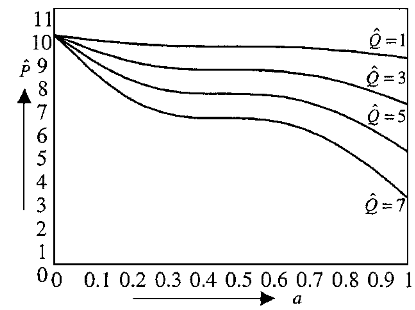

Fig. 3 Critical buckling load as a function of intermediate load and its position.

IV. Higher-Order Approximation

The method can be extended to produce a higher-order approximation as follows. Taylor series expansion of order p gives

$$\boldsymbol{A}\left(\omega^{(0)} + \varepsilon\right) = \boldsymbol{A}\left(\omega^{(0)}\right) + \sum_{k=1}^{p} \frac{\varepsilon^k}{k!} \frac{\mathrm{d}^k \boldsymbol{A}}{\mathrm{d}\omega^k}\bigg|_{\omega = \omega^{(0)}} \tag{37}$$

where

$$\boldsymbol{A}_k = \frac{1}{k!} \frac{\mathrm{d}^k \boldsymbol{A}}{\mathrm{d}\omega^k}, \qquad k = 1, 2, \ldots, p \tag{38}$$

Hence,

$$\det\left[\boldsymbol{A}\left(\omega^{(0)} + \varepsilon\right)\right] = 0 \tag{39}$$

implies

$$\det\left[\boldsymbol{A}\left(\omega^{(0)}\right) + \varepsilon \boldsymbol{A}_1\left(\omega^{(0)}\right) + \cdots + \varepsilon^p \boldsymbol{A}_p\left(\omega^{(0)}\right)\right] = 0 \tag{40}$$

The iterative correction for ε can, thus, be determined by the least eigenvalue (in absolute value sense) of the pth-order eigenvalue problem

$$\left[\boldsymbol{A}\left(\omega^{(0)}\right) + \lambda \boldsymbol{A}_1\left(\omega^{(0)}\right) + \cdots + \lambda^p \boldsymbol{A}_p\left(\omega^{(0)}\right)\right]\boldsymbol{z} = 0 \tag{41}$$

A first-order realization of Eq. (41) is given by

$$[\boldsymbol{G}(\omega) - \lambda \boldsymbol{H}(\omega)]\boldsymbol{y} = \boldsymbol{0} \tag{42}$$

where

$$\boldsymbol{G}(\omega) = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n & & \\ & \boldsymbol{0} & \ddots & \\ & & \ddots & \boldsymbol{I}_n \\ -\boldsymbol{A}(\omega) & -\boldsymbol{A}_1(\omega) & \cdots & -\boldsymbol{A}_{p-1}(\omega) \end{bmatrix} \tag{43}$$

Table 3 High-order approximation of the natural frequencies of the rod

Initial guess	ω_i	First order	Second order
0.1	1.8980	14	6
3.2	4.3437	10	5
6.4	8.13909	13	6
9.6	10.58616	6	5

$$H(\omega) = \begin{bmatrix} I_n & & & \\ & I_n & & \\ & & \ddots & \\ & & & A_p(\omega) \end{bmatrix} \tag{44}$$

$$y = \begin{pmatrix} z \\ \lambda z \\ \vdots \\ \lambda^{p-1} z \end{pmatrix} \tag{45}$$

and I_n is the identity matrix of dimension n.

To demonstrate the use of high-order approximation, we solved the axially vibrating rod example again, using an approximation of order $p = 2$. The results obtained are shown in Table 3. As may be expected, the number of iterations required for convergence has been reduced. In addition, it is clear from a theoretical point of view that the convergence interval associated with higher-order approximation is larger than that obtained from low-order approximation. Therefore, convergence may be achieved by using an inferior initial guess.

V. Quadratic Eigenvalue Problem

Note that the method developed in the preceding sections is quite general and holds for matrices with nonlinear elements that are not necessarily transcendental. Consider for example the quadratic eigenvalue problem associated with finite dimensional vibrating systems:

$$A(\lambda) = \lambda^2 M + \lambda C + K \tag{46}$$

where M, C, and K are symmetric. The algorithm of Sec. II can be applied to the problem using

$$B(\lambda) = -\frac{dA}{d\lambda} = -2\lambda M - C \tag{47}$$

The standard method for solving such problems involves transforming the equations into first-order form. The transformation requires doubling the dimensions of the problem and destroying the symmetry of the system. With the method developed here, the matrix size and the symmetry of the system is unaltered, as demonstrated by the following example.

The spectrum of a discrete model of a vibrating system is given as follows. The equation of motion for the system shown in Fig. 4 is given by Eq. (49) with

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$K = \begin{bmatrix} 5 & -2 & -1 \\ -2 & 4 & -2 \\ -1 & -2 & 3 \end{bmatrix} \tag{48}$$

We define

$$A(\lambda) = \begin{bmatrix} \lambda^2 + \lambda + 5 & -2 & -\lambda - 1 \\ -2 & \lambda^2 + 3\lambda + 4 & -\lambda - 2 \\ -\lambda - 1 & -\lambda - 2 & \lambda^2 + 2\lambda + 3 \end{bmatrix} \tag{49}$$

Table 4 Eigenvalues of Eqs. (46) and (48)

Initial guess	Number of iteration	Calculated λ_i
$-1-2i$	5	$-0.7226-2.2521i$
$-1+2i$	5	$-0.7226+2.2521i$
$-1-i$	6	$-0.4863-0.6063i$
$-1+i$	6	$-0.4863+0.6063i$
$-2-i$	5	$-1.7911-1.2358i$
$-2+i$	5	$-1.7911+1.2358i$

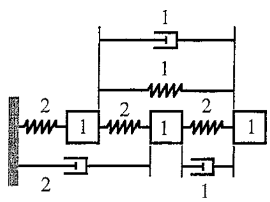

Fig. 4 Mass–spring–damper system.

and obtain from Eq. (47)

$$B(\lambda) = \begin{bmatrix} -2\lambda - 1 & 0 & 1 \\ 0 & -2\lambda - 3 & 1 \\ 1 & 1 & -2\lambda - 2 \end{bmatrix} \tag{50}$$

The Newton's eigenvalue method has been applied to the problem yielding the eigenvalues shown in Table 4. The initial guess used and the number of iteration required for convergence with a tolerance of 10^{-12} are also shown in Table 4. Note that the eigenvalues has been determined by using 3×3 symmetric matrices, whereas the standard method of solving this problem involves a first-order realization with matrices of dimension 6×6.

Higher-order approximation may also be applied to the quadratic eigenvalue problem (46), yielding, via Eq. (37),

$$A(\lambda_0 + \varepsilon) = A(\lambda_0) + \varepsilon A_1(\lambda_0) + \varepsilon^2 A_2(\lambda_0) \tag{51}$$

where

$$A = \lambda^2 M + \lambda C + K \tag{52}$$

$$A_1 = 2\lambda M + C \tag{53}$$

$$A_2 = M \tag{54}$$

However, a first-order realization leads via Eqs. (42–44) to an eigenvalue problem of dimension $2n$ that is of the same complexity as the original problem.

VI. Spectrum Estimation of a Nonuniform Distributed Parameter System

The main application of the transcendental eigenvalue problem is in estimating the spectrum of continuously varying distributed parameter systems. Consider, for example, an axially vibrating rod of unit length with cross-sectional area $A(x) = e^x$, modulus of elasticity E, and density ρ, which is fixed at $x = 0$ and free to oscillate at the other end $x = 1$. The system is governed by the differential equation

$$\frac{\partial}{\partial x}\left[E e^x \frac{\partial u(x,t)}{\partial x}\right] = \rho e^x \frac{\partial^2 u(x,t)}{\partial t^2}, \qquad 0 < x < 1, \qquad t > 0 \tag{55}$$

which leads after separation of variables to the following eigenvalue problem:

$$v'' + v' + \eta v = 0, \qquad \eta = \omega^2(\rho/E), \qquad v(0) = 0, \qquad v'(1) = 0 \tag{56}$$

The eigenvalues of the system (56) are the roots of the frequency equation[3]:

$$\tan\sqrt{\eta - \tfrac{1}{4}} = 2\sqrt{\eta - \tfrac{1}{4}} \tag{57}$$

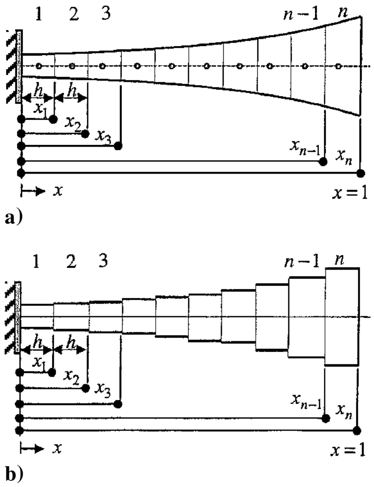

Fig. 5 Exponential rod: a) discrete model of the order *n* and b) approximation using a rod with piecewise uniform cross-sectional area.

Finite difference and finite element methods are widely used in approximating the eigenvalues of distributed parameter systems. Note, however, that finite element and finite difference methods produce fair estimation only to the low part of the spectrum. One can expect from a discrete model of order n to produce fewer than $n/3$ eigenvalues with reasonable accuracy (see, e.g., Refs. 4–6). In contrast, the distributed parameter system has infinitely many eigenvalues. To demonstrate this phenomenon, we replace the given continuous system by an equivalent discrete model of order n with uniform element length $h = 1/n$ as shown in Fig. 5a. The eigenvalue problem corresponding to the discrete model of the rod is

$$(\boldsymbol{K} - \eta \boldsymbol{M})\boldsymbol{v} = \boldsymbol{0} \tag{58}$$

$$\boldsymbol{V} = \begin{bmatrix} \gamma_{11} & -\sigma_{11} & -\gamma_{12} & \sigma_{12} & & & & \\ 0 & 0 & \gamma_{22} & -\sigma_{22} & -\gamma_{23} & \sigma_{23} & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \gamma_{n-2,n-2} & -\sigma_{n-2,n-2} & -\gamma_{n-2,n-1} & \sigma_{n-2,n-1} & & \\ & & & & \gamma_{n-1,n-1} & -\sigma_{n-1,n-1} & -\gamma_{n-1,n} & \sigma_{n-1,n} \end{bmatrix} \tag{66}$$

A finite difference model using central derivatives leads to the following stiffness and mass matrices:

$$\boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & & & & \\ -k_2 & k_2 + k_3 & -k_3 & & & \\ & -k_3 & \ddots & \ddots & & \\ & & \ddots & \ddots & -k_{n-1} & \\ & & & -k_{n-1} & k_{n-1} + k_n & -k_n \\ & & & & -k_n & k_n \end{bmatrix} \tag{59}$$

$$\boldsymbol{M} = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{bmatrix} \tag{60}$$

where

$$k_i = (EA)_i/h, \qquad m_i = (\rho A)_i h, \qquad i = 1, 2, \ldots, n \tag{61}$$

Finite element modeling using piecewise constant elements with linear shape functions leads, similarly, to the generalized eigenvalue problem (58) with the same stiffness matrix given by Eq. (59) and a mass matrix of the form

$$\boldsymbol{M} = \frac{1}{6}\begin{bmatrix} 2m_1 + 2m_2 & m_2 & & & \\ m_2 & 2m_2 + 2m_3 & m_3 & & \\ & \ddots & \ddots & \ddots & \\ & & m_{n-1} & 2m_{n-1} + 2m_n & m_n \\ & & & m_n & 2m_n \end{bmatrix} \tag{62}$$

There have been 20 eigenvalues of the exponential rod estimated using finite difference and finite elements models of order $n = 20$. The exact eigenvalues have been calculated using a bisection method applied to the frequency equation (57). The results are shown in Fig. 6a. It is apparent from Fig. 6a that only about six eigenvalues have been estimated accurately by these methods.

The exponential rod as shown in Fig. 5b can also be formulated in a transcendental eigenvalue form (1) with

$$\boldsymbol{A}(\eta) = \begin{bmatrix} \boldsymbol{p}^T \\ \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{q}^T \end{bmatrix} \tag{63}$$

where

$$\boldsymbol{p} = (0 \quad 1 \quad \cdots \quad 0 \quad 0)^T \tag{64}$$

$$\boldsymbol{U} = \begin{bmatrix} \alpha_1 & \beta_1 & -\alpha_1 & -\beta_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 & \beta_2 & -\alpha_2 & -\beta_2 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & \alpha_{n-2} & \beta_{n-2} & -\alpha_{n-2} & -\beta_{n-2} & 0 & 0 \\ & & & & \alpha_{n-1} & \beta_{n-1} & -\alpha_{n-1} & -\beta_{n-1} \end{bmatrix} \tag{65}$$

$$\boldsymbol{q} = \left(0 \quad \cdots \quad 0 \quad \sqrt{\eta}\cos\sqrt{\eta}L \quad -\sqrt{\eta}\sin\sqrt{\eta}L\right)^T \tag{67}$$

and where

$$\alpha_i = \sin\sqrt{\eta}x_i \tag{68}$$

$$\beta_i = \cos\sqrt{\eta}x_i \tag{69}$$

$$\gamma_{ij} = A_j\sqrt{\eta}\cos\sqrt{\eta}x_i \tag{70}$$

$$\sigma_{ij} = A_j\sqrt{\eta}\sin\sqrt{\eta}x_i \tag{71}$$

The matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ impose the matching conditions of displacement and force, respectively, and the vector $\boldsymbol{p}$ expresses the free-end condition in the rod. The variable A_j denotes the cross-sectional area of the jth element. The problem can then be solved using the algorithm developed in Secs. II and III. We solved this problem using a small model order of $n = 3$, where the rod is divided into three uniform portions of cross-sectional areas

$$A_i = \begin{cases} \left(e^{\frac{1}{3}} + e^{0}\right)/2, & i = 1 \\ \left(e^{\frac{2}{3}} + e^{\frac{1}{3}}\right)/2, & i = 2 \\ \left(e^{1} + e^{\frac{2}{3}}\right)/2, & i = 3 \end{cases} \tag{72}$$

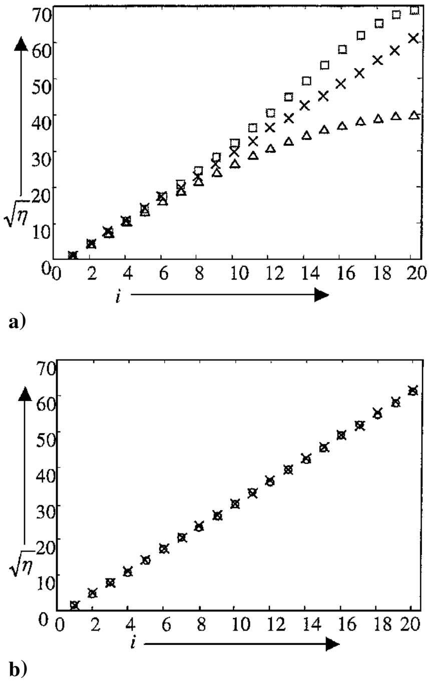

Fig. 6 Natural frequencies of exponential rod: a) comparison between the ×, exact solution; △, finite difference; and □, finite element models of order $n = 20$ and b) comparison between the ×, exact solution and ○, transcendental eigenvalue model of order $n = 3$.

The matrix $A(\eta)$ in Eq. (63) with $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, and $L = 1$ takes the form

$$
\boldsymbol{A}(\eta) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \sin\xi & \cos\xi & -\sin\xi & -\cos\xi & 0 & 0 \\ 0 & 0 & \sin\psi & \cos\psi & -\sin\psi & -\cos\psi \\ A_1\cos\xi & -A_1\sin\xi & -A_2\cos\xi & A_2\sin\xi & 0 & 0 \\ 0 & 0 & A_2\cos\psi & -A_2\sin\psi & -A_3\cos\psi & A_3\sin\eta \\ 0 & 0 & 0 & 0 & \cos 3\xi & -\sin 3\xi \end{bmatrix} \tag{73}
$$

where $\xi = \frac{1}{3}\sqrt{\eta}$ and $\psi = \frac{2}{3}\sqrt{\eta}$

We have determined the 20 lowest roots of $\det[\boldsymbol{A}(\eta)] = 0$ by using the method described in Sec. II. The results are shown in Fig. 6b together with the exact analytical solution. It is clearly demonstrated that the transcendental model has produced good approximation for all of the eigenvalues obtained. Note that these results have been found using a matrix of order 6×6, whereas the inferior results obtained by the finite difference and finite element models have been produced using matrices of higher dimensions, 20×20. Note also that the transcendental model (73), with its small dimensional matrix, can produce unlimited number of eigenvalues, whereas finite elements and finite difference models of order n can at most produce n eigenvalues, of which only about $n/3$ have fair accuracy. This problem may be solved alternatively by using Gaussian elimination as in Lake and Mikulas[7] or by the method suggested by Williams and Kennedy.[8]

VII. Conclusions

Some problems in vibration and structural stability of distributed parameter systems lead to transcendental eigenvalue problems involving matrices with transcendental elements. The associated discrete models are characterized by algebraic eigenvalue problems with matrices of constant entries. Finite element or finite difference methods implement discrete models in approximating the eigenvalues of distributed parameters systems. It has been noted (see, for example, Refs. 4–6) that the asymptotic behavior of a continuous system is different from that associated with its representing discrete model. This discrepancy in behavior originates from a transcendental function lacking the ability to be properly approximated by an nth degree polynomial over a range including n neighboring roots. Consequently, a large-order finite difference or finite element models can approximate accurately only small number of eigenvalues. The method developed here is a different approach to overcome the same difficulties.

In theory, the determinant of the transcendental matrix can be expanded by using its fundamental definition. This is, however, prohibited by numerical considerations. Expanding a determinant of a matrix of dimension n involves $n!$ floating point operations, that is, additions and multiplications, which may hamper the calculation of a relative small problem of dimension $n > 10$, even when fast computers are used. Determinant expansion using Gaussian elimination, as suggested by Lake and Mikulas,[7] is more economical computationally. It requires, however, symbolic manipulations. With the Newton's eigenvalue iteration method, the difficulties associated with evaluating a determinant of a transcendental matrix is circumvented. Williams and Kennedy[8] have developed an alternative multiple determinant parabolic interpolation method to solve the transcendental eigenvalue problem.

In application, a distributed parameters system with continuously varying parameters may be approximated by another distributed parameter system of piecewise uniform properties. The transcendental eigenvalue problem obtained can then be applied effectively to solve the problem. With this method, a large number of eigenvalues can be obtained by solving a transcendental problem of small dimension. This idea has been demonstrated in Sec. VI, where accurate approximations of a large number of eigenvalues corresponding to a nonuniform vibrating rod have been obtained by the use of a small transcendental model.

Acknowledgments

The work was supported in part by National Science Foundation Grant CMS-9978786. The authors thank an anonymous reviewer for drawing their attention to the seminal work of W. H. Yang in Ref. 1.

References

[1]Yang, W. H., "A Method for Eigenvalues of the Sparse λ-Matrices," *International Journal for Numerical Methods in Engineering*, Vol. 19, No. 6, 1983, pp. 943–948.

[2]Timoshenko, S. P., and Gere, J. M., "Elastic Buckling of Bars and Frames," *Theory of Elastic Stability*, 2nd ed., McGraw–Hill, New York, 1961, pp. 98–100.

[3]Ram, Y. M., "An Inverse Mode Problem for the Continuous Model of an Axially Vibrating Rod," *Journal of Applied Mechanics*, Vol. 61, No. 3, 1994, pp. 624–628.

[4]Boley, D., and Golub, G. H., "A Survey of Matrix Inverse Eigenvalue Problems," *Inverse Problems*, Vol. 3, No. 4, 1987, pp. 595–622.

[5]Paine, J., "A Numerical Method for the Inverse Sturm–Liouville Problem," *SIAM Journal on Scientific and Statistical Computing*, Vol. 5, No. 1, 1984, pp. 149–156.

[6]Paine, J. W., de Hoog, F., and Anderssen, R. S., "On the Correction of Finite Difference Eigenvalue Approximations for Sturm–Liouville Problems," *Computing*, Vol. 26, No. 2, 1981, pp. 123–139.

[7]Lake, M. S., and Mikulas, M. M., "Buckling and Vibration Analysis of a Simply Supported Column with a Piecewise Constant Cross Section," NASA TP-3090, March 1991.

[8]Williams, F. W., and Kennedy, D., "Reliable use of Determinants to Solve Non-Linear Structural Eigenvalue Problems Efficiently," *International Journal for Numerical Methods in Engineering*, Vol. 26, No. 8, 1988, pp. 1825–1841.

A. Berman
Associate Editor